FUTURE TEACHERS' USE OF MULTIPLICATION AND FRACTIONS WHEN EXPRESSING PROPORTIONAL RELATIONSHIPS

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ICME-13
Hamburg, Germany
July 27, 2016

PRoMPT: Proportional Reasoning of Middle Grades Pre-service Teachers

InPReP2: Investigating Proportional Relationships from Two Perspectives

Project Website:

http://temrrg.wix.com/temrrg

Introduction

- Core concepts of elementary and secondary mathematics education (NCTM, 2000),
- Proportional relationships provide solid foundation for diverse topics such as linear functions, slope, geometric similarity, probability,
- A large body of research on fractions, ratios, and proportions (e.g., Behr, Harel, Post, & Lesh, 1992) has demonstrated that these topics are some of the most challenging for students to learn, but are also critically important.

Background

- Teachers have similar difficulties as students in reasoning about proportional relationships with quantities,
- Teachers can resort to cross-multiplication as a rote computation procedure and can guess at operations (e.g., Harel & Behr, 1995); can have difficulty coordinating two proportionally related quantities (e.g., Orrill & Brown, 2012), and can rely on additive relationships rather than multiplicative ones when reasoning about these quantities (e.g., Ölmez, 2016).
- Existing studies have acknowledged that solving proportions involving whole-number multiples than those involving fraction multiples is easier (e.g., Kaput & West, 1994), but they have not examined how quantitative definitions for multiplication and for fractions can support and constrain reasoning about proportional relationships.

Theoretical Framework

Definition of Multiplication (Beckmann & Izsak, 2015)

$$M \cdot N = P$$

Multiplier: number of equal groups

Multiplicand: number of units in 1 group

Product: number of units in M groups

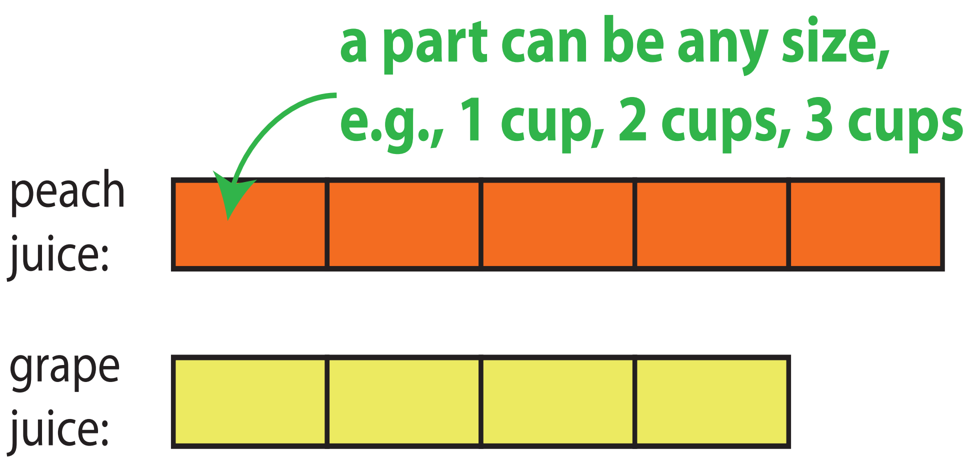
Definition for Fractions (CCSS; Common Core State Standards Initiative, 2010)

(1) 1/*b* is the quantity formed by one part when a unit amount (or whole) is divided into *b* equal parts; each part is 1/*b* of the unit amount.

(2) *a*/*b* is the quantity formed by *a* parts of size 1/*b* of the unit amount.

Variable Parts Perspective

Quantities of peach and grape juice in a mixture with 5 to 4 ratio :

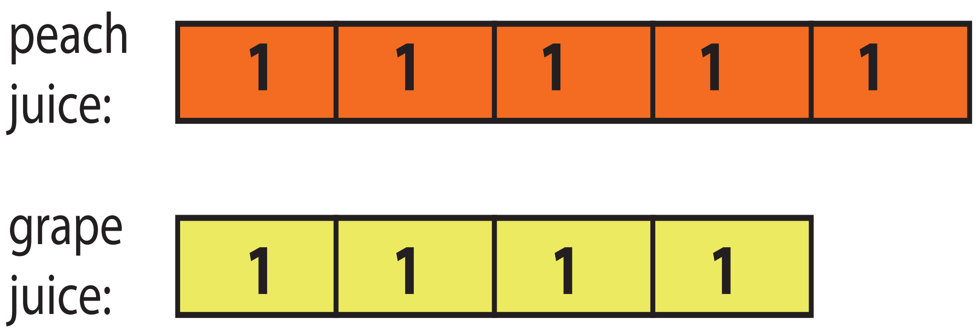


cups per part	cups peach	cups grape	cups total
1	5	4	9
2	10	8	18
3	15	12	27

Variable Parts Perspective

Quantities of peach and grape juice in a mixture with 5 to 4 ratio :

(**5 parts**)•(**1** cups per part)



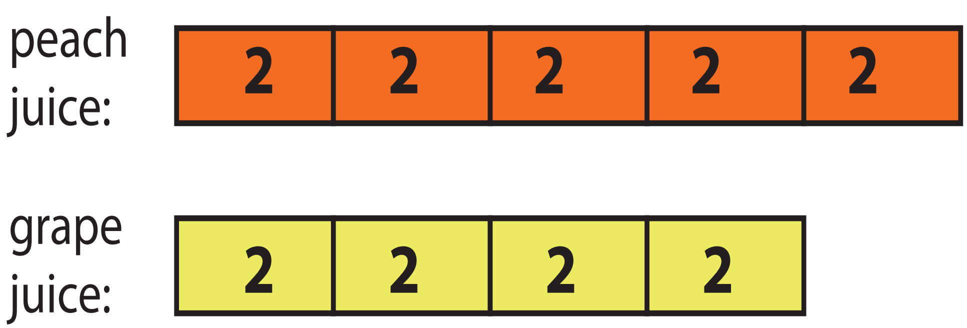
(**4 parts**)•(**1** cups per part)

cups per part	cups peach	cups grape	cups total
1	**5**	**4**	**9**
2	10	8	18
3	15	12	27

Variable Parts Perspective

Quantities of peach and grape juice in a mixture with
5 to 4 ratio :

(**5 parts**)•(**2** cups per part)



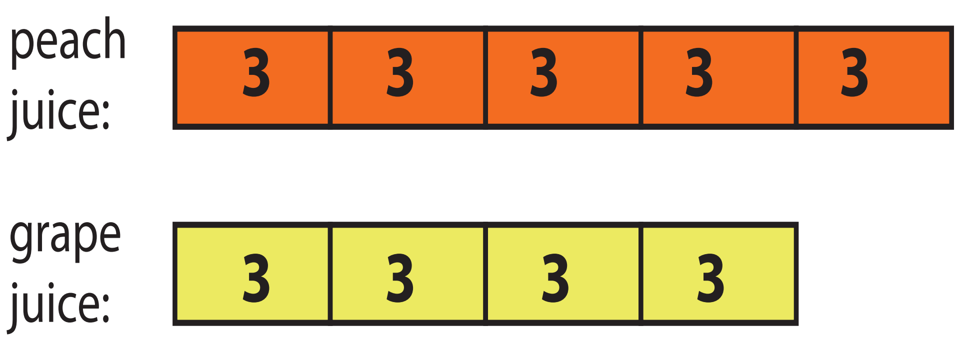
(**4 parts**)•(**2** cups per part)

cups per part	cups peach	cups grape	cups total
1	5	4	9
2	**10**	**8**	**18**
3	15	12	27

Variable Parts Perspective

Quantities of peach and grape juice in a mixture with 5 to 4 ratio :

(**5 parts**)•(**3** cups per part)

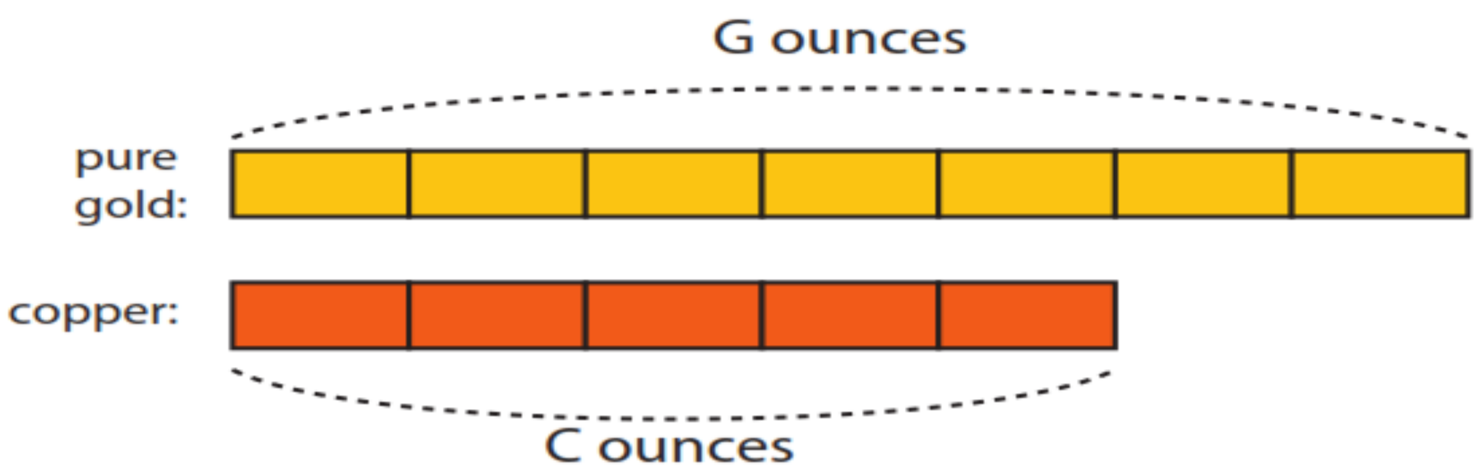


(**4 parts**)•(**3** cups per part)

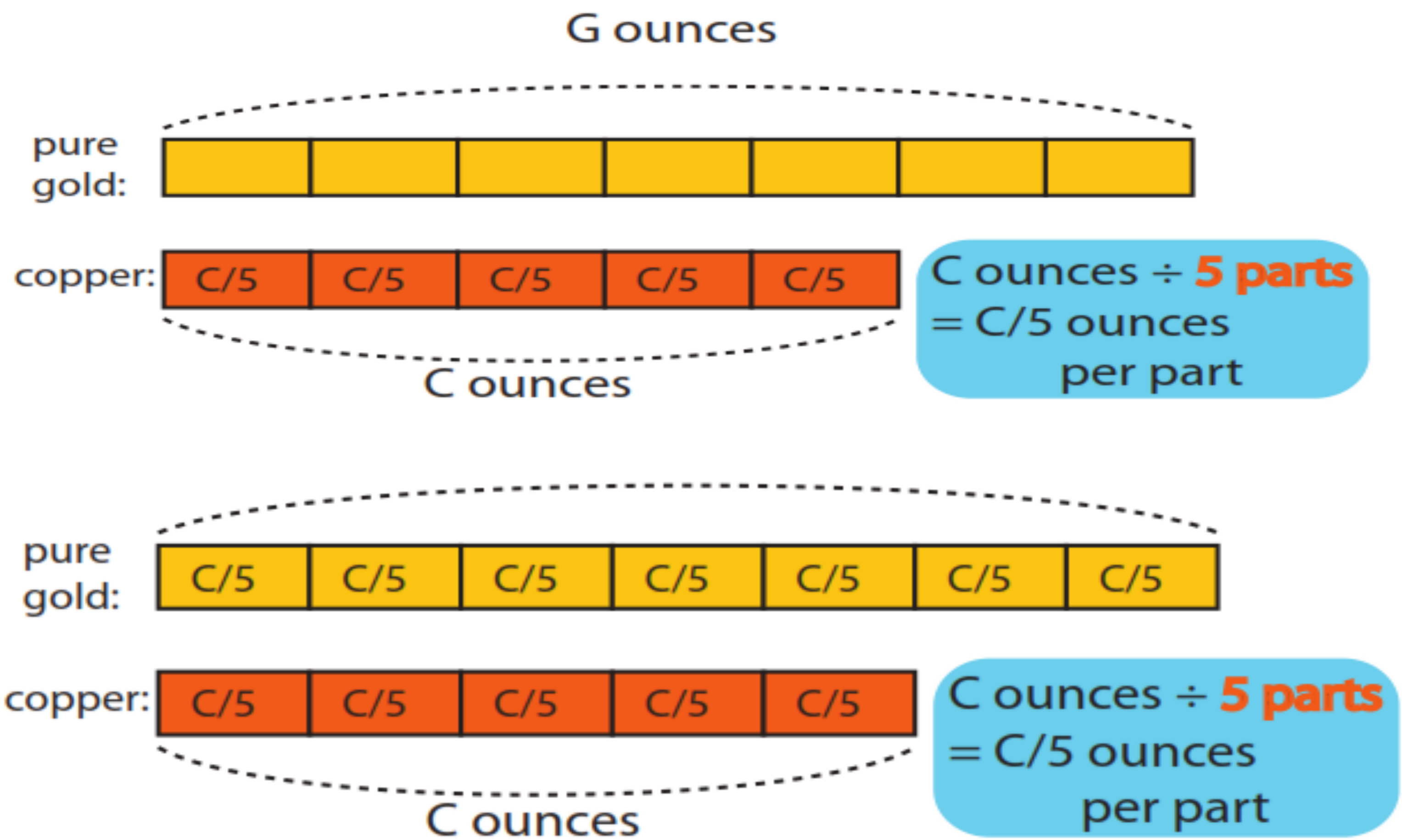
cups per part	cups peach	cups grape	cups total
1	5	4	9
2	10	8	18
3	**15**	**12**	**27**

Illustration: The Jewelry problem

- A company makes jewelry using gold and copper in a 7 to 5 ratio. Let *G* and *C* be some unspecified number of ounces of gold and copper the company will use that same day. Please use a strip diagram to help you explain the relationship between *G* and *C*.

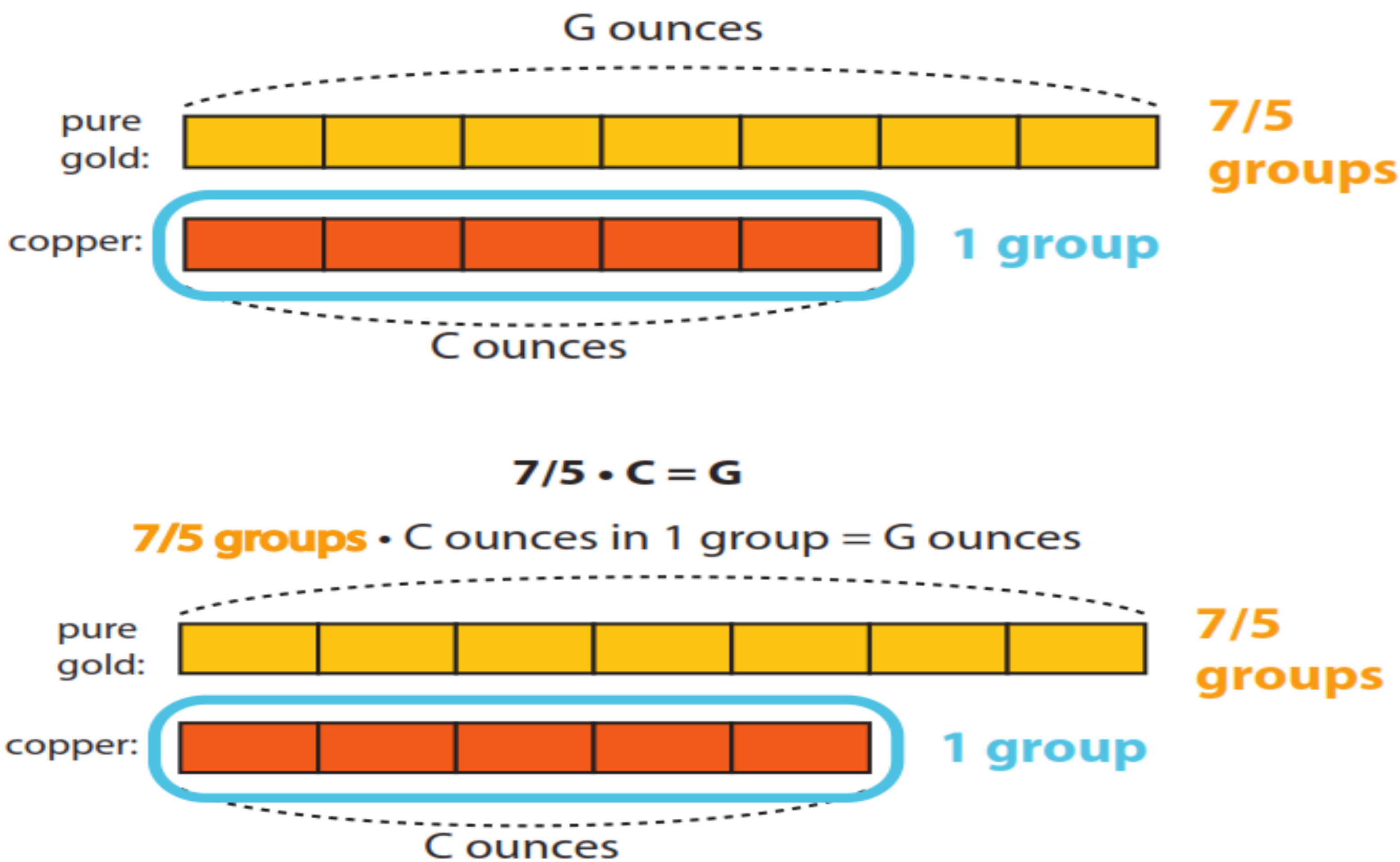


1st solution: Definition of Multiplication



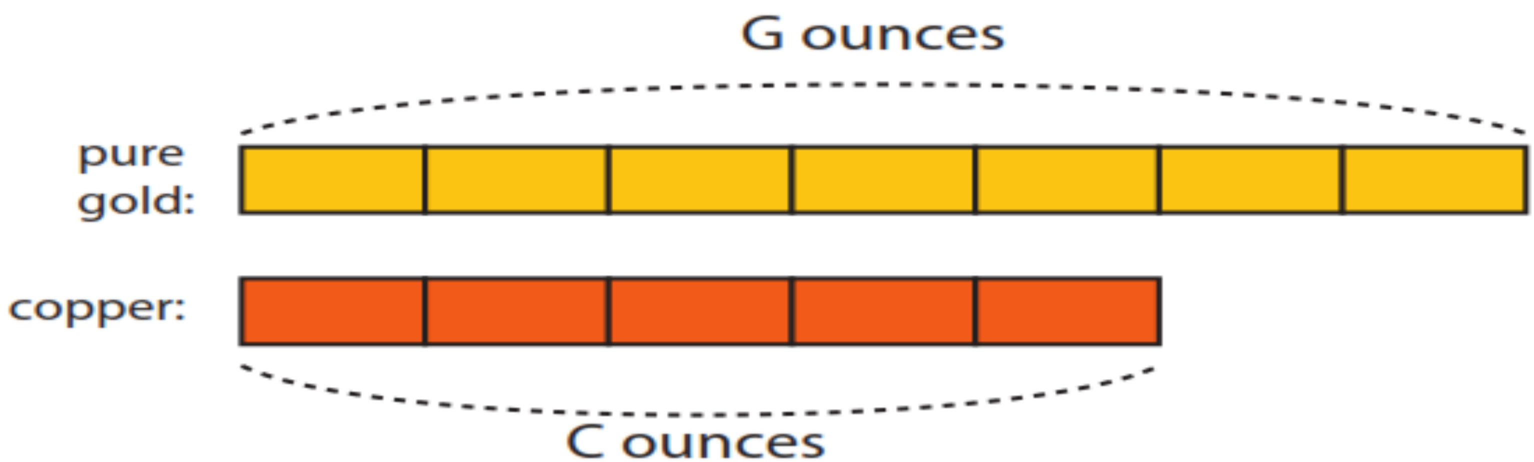
7 (groups) • C/5 (ounces in one group) = G (ounces in 7 groups)

2nd solution: Definition of Multiplication



7/5 (groups) • C (ounces in one group) = G (ounces in 7/5 groups

Definition for Fractions



- 7/5 can be seen as “7 parts each of size 1/5 of the copper”
- 5/7 can be seen as “5 parts each of size 1/7 of the gold”

Purpose & Research Questions

- To examine how 6 future middle grades mathematics teachers express explicit, quantitative definitions for multiplication and for fractions when reasoning about proportional relationships.
- How do future teachers reason with these definitions for multiplication and for fractions when solving proportion problems?

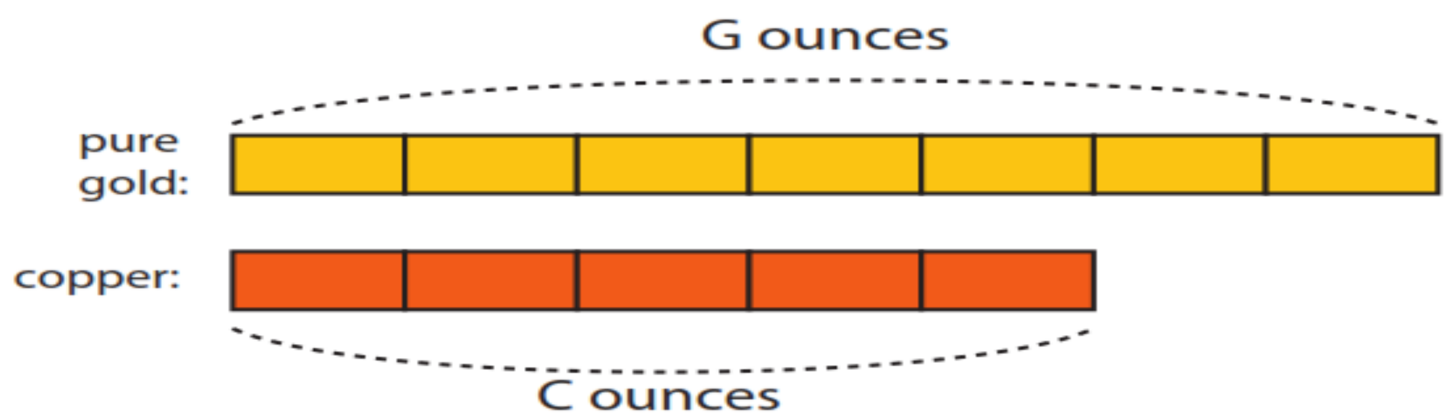
Methods

- A larger, ongoing project of future middle grades (grades 4-8) mathematics teachers' ecology of multiplicative reasoning.
- Fall 2014 & Spring 2015: the definitions for multiplication and for fractions, the variable parts perspective on proportional relationships including strip diagrams (Beckmann & Izsák, 2015).
- 6 future teachers were recruited based on their performance on a fractions survey that focused on multiplication and division of fractions in terms of quantities.
- Each of these six teachers participated in a total of six individual cognitive interviews during the two semesters in 2014-2015.
- Data for the present study consist of future teachers' interview videos, trancripts, and written work within two tasks that were given during their fourth interview.
- Students had received initial instruction on ratios and proportional relationships but not yet developed equations in two variables.

Interview Tasks

- Task 1: How do you interpret the meaning of 1/6 • X ?

- Task 2: Jewelry Problem: A company makes jewelry using gold and copper in a 7 to 5 ratio. Let *G* and *C* be some unspecified number of ounces of gold and copper the company will use that same day. Please use a strip diagram to help you explain the relationship between *G* and *C*.



Results

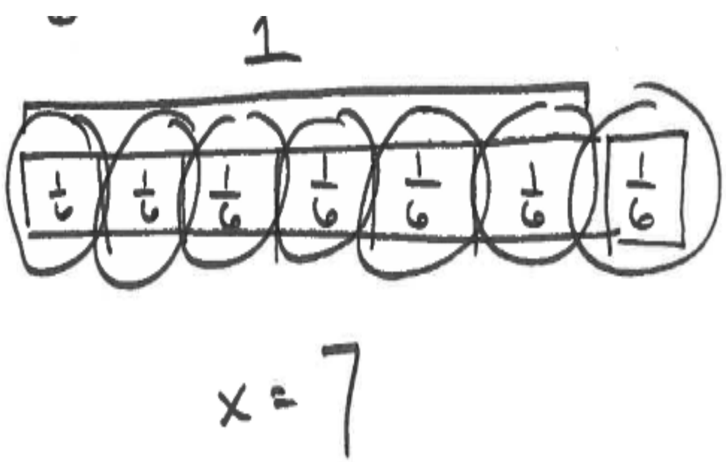
Names	Definition of multiplication	Definition of fractions	Generating a correct equation
Alex	NO	Not enough evidence	NO
Jeff	NO	Not enough evidence	With prompting
Linda	YES	YES	YES
Claire	YES	YES	YES
Diana	NO	YES	NO
Kelly	YES	Not enough evidence	With prompting

Table 1: A summary table presenting the future teachers' performances in the two tasks.

- Future teachers who used both definitions for multiplication and for fractions seemed to generate a correct equation about proportional relationships.

Results_Task 1

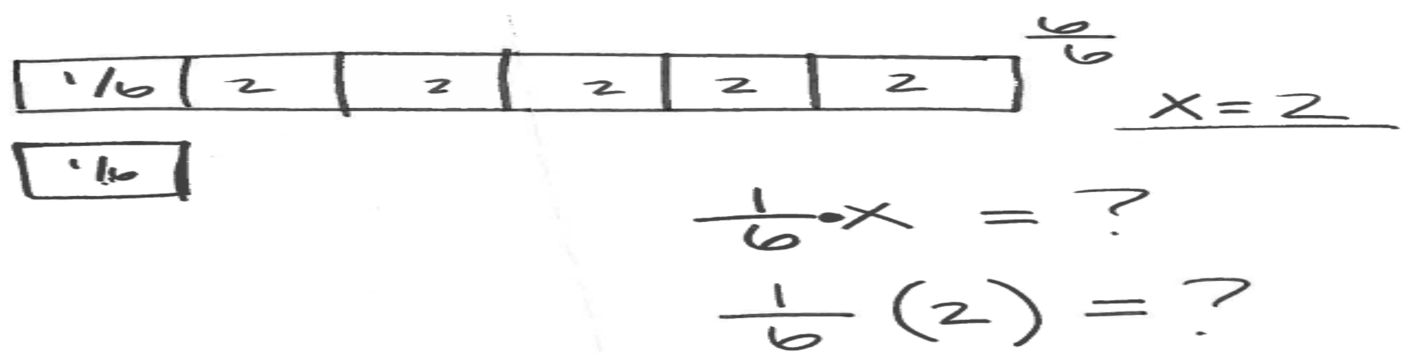
- In response to Task 1, half of the future teachers (Jeff, Diana, Alex) did not use the definition of multiplication consistently.
- Jeff and Diana could interpret 4 • *X* using the definition of multiplication (i.e., "4 groups with *X* in each group") but could not interpret 1/6 as a number of groups.
- Jeff switched the order of the given multiplication, turning 1/6 from the multiplier position into the multiplicand, as indicated also in his strip diagram:



- Jeff: "I see *X* as the whole 7 pieces of 1/6, because if we are saying *X* equals 7, so we have 7 groups of 1/6."

Results_Task 1

- Alex did use the definition of multiplication, but her referent units lacked the precision and her use of it was not consistent across the interview. She interpreted the 1 group of the multiplicand as 1 part of the whole strip instead of 6 parts of the whole strip.
- Alex also labeled some parts of her strip diagram with 1/6 and others with 2 as a specific value for X.



Results_Task 1

Names	Definition of multiplication	Definition of fractions	Generating a correct equation
Alex	NO	Not enough evidence	NO
Jeff	NO	Not enough evidence	With prompting
Linda	YES	YES	YES
Claire	YES	YES	YES
Diana	NO	YES	NO
Kelly	YES	Not enough evidence	With prompting

Table 1: A summary table presenting the future teachers' performances in the two tasks.

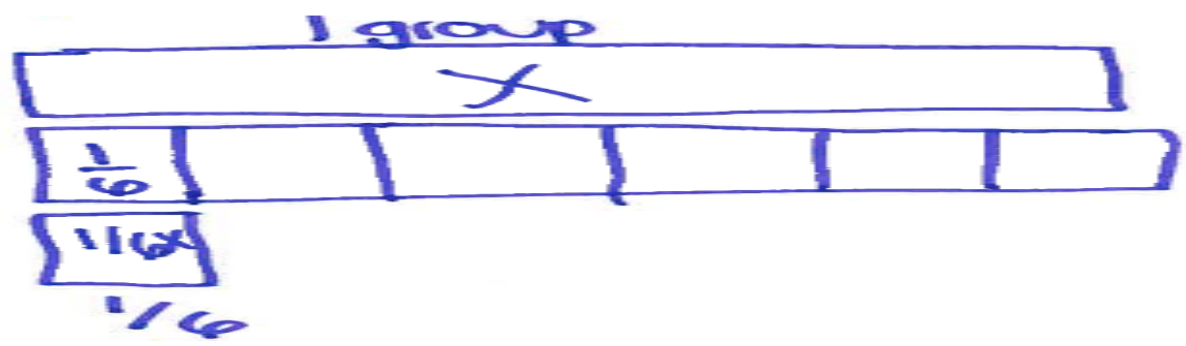
- Another half of the future teachers (Linda, Claire, Kelly) demonstrated a solid understanding of the definition of multiplication by coordinating the referent units for each term in the equation of $1/6 \cdot X$.

Results_Task 1

Kelly: "We have 1/6th of a group. And in that group it's size X. And then the product will be the total size of 1/6th of a group."

Int. : "Could you give like a context and or drawing to explain that?"

Kelly: "Okay (draws Figure 1b). The size of one group is X, so there is X in one group but we want to know how many of X is in 1/6th of a group. So if I take this one group, divide it into 6, each part is one part of 6 total parts, each part of 1/6th in size. And if I just want to look at this 1/6th part of my one group, there will be 1/6th of X inside it."



Results_Task 2

Names	Definition of multiplication	Definition of fractions	Generating a correct equation
Alex	NO	Not enough evidence	NO
Jeff	NO	Not enough evidence	With prompting
Linda	YES	YES	YES
Claire	YES	YES	YES
Diana	NO	YES	NO
Kelly	YES	Not enough evidence	With prompting

Table 1: A summary table presenting the future teachers' performances in the two tasks.

- Alex and Diana could not generate an equation involving G and C.
- Diana produced C = 5 • 1/7 and interpreted the equation using the definition for fractions (i.e.,"5 parts each of size 1/7 of G ounces").

Results_Task 2

- Alex produced 5 • *C*/5 = ? for copper and 7 • *G*/7 = ? for gold, but she could not form an equation that involves *G* and *C* using the definition of multiplication.
- Jeff and Kelly were able to produce a correct equation with prompting. Kelly interpreted both of her expressions, 5 • *G*/7 and 5/7 • *G*, using the definition of multiplication.
- For 5/7 • *G*, Kelly was able to visualize 5/7 as "the number of groups" from the strip diagram and explained "5/7ths of *G* makes copper."
- Jeff, switched his equation from *G* • 5/7 = *C* into 5 • *G*/7 = *C* and then used the definition of multiplication (i.e., "5 groups times the ounces per group gives the ounces in 5 groups").

Results_Task 2

- Linda and Claire succeeded in generating quickly a single correct equation with a fraction multiplier as $C = 5/7 \bullet G$ by visualizing 5/7 as "the number of groups" in their strip diagram drawings.
- Claire interpreted her equation, $5/7 \bullet G = C$, from both the definition for fractions (i.e., "5 parts each size 1/7 of the whole amount of gold") and the definition of multiplication (i.e., "5/7 groups times the amount of ounces in 1 group is equal to the amount of copper in 5/7 groups of gold") by keeping the language of each definition distinct.

Conclusion

Names	Definition of multiplication	Definition of fractions	Generating a correct equation
Alex	NO	Not enough evidence	NO
Jeff	NO	Not enough evidence	With prompting
Linda	YES	YES	YES
Claire	YES	YES	YES
Diana	NO	YES	NO
Kelly	YES	Not enough evidence	With prompting

Table 1: A summary table presenting the future teachers' performances in the two tasks.

- Three of the six teachers (Alex, Jeff, Diana) experienced difficulties distinguishing the roles played by the multiplier and multiplicand.
- Specifically, they had trouble identifying the multiplier as the number of groups when multipliers were fractions, and this even caused some of them to switch the order of their equations.
- Moreover, the placement of fractions as multipliers in the tasks caused these teachers to misuse phrasing from the definition for fractions when applying the definition of multiplication.

Conclusion

Names	Definition of multiplication	Definition of fractions	Generating a correct equation
Alex	NO	Not enough evidence	NO
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Linda	YES	YES	YES
Claire	YES	YES	YES
Diana	NO	YES	NO
Kelly	YES	Not enough evidence	With prompting

Table 1: A summary table presenting the future teachers' performances in the two tasks.

- Of those three teachers (Linda, Claire, Kelly) who were able to use the definition of multiplication, all were able to keep distinct wording from the definitions for multiplication and for fractions.
- Their use of the definitions for multiplication and for fractions seemed to facilitate their generation and explanation of equations.
- Furthermore, when multipliers were placed as fractions in the tasks, these future teachers were also able to identify the multiplier as the number of groups by viewing how many groups (or parts) of one strip were in another strip in their drawings.

Conclusion

- Future teachers who did not have explicit, quantitative definitions for multiplication and for fractions had difficulties in reasoning about proportional relationships.
- Robust arguments about proportional relationships and generating equations depend in turn on the ability to keep distinct the definitions for multiplication and for fractions.
- In addition to stating the definition of multiplication verbally, future teachers need to visualize how many parts of one strip are nested in another strip by viewing the relationships between the multiplier and multiplicand in the strip diagrams.

References

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